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THERMODYNAMIC FORMALISM FOR NULL RECURRENT POTENTIALS

BY

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ABSTRACT

We extend Ruelle's Perron–Frobenius theorem to the case of Hölder continuous functions on a topologically mixing topological Markov shift with a countable number of states. Let $P(\phi)$ denote the Gurevic pressure of ϕ and let L_ϕ be the corresponding Ruelle operator. We present a necessary and sufficient condition for the existence of a conservative measure ν and a continuous function h such that $L_\phi^*\nu = e^{P(\phi)}\nu$, $L_\phi h = e^{P(\phi)}h$ and characterize the case when $\int h d\nu < \infty$. In the case when $dm = hd\nu$ is infinite, we discuss the asymptotic behaviour of L_ϕ^k, and show how to interpret dm as an equilibrium measure. We show how the above properties reflect in the behaviour of a suitable dynamical zeta function. These results extend the results of [18] where the case $\int hd\nu < \infty$ was studied.

1. Introduction and statement of main results

Let S be a countable set of **states** and $\mathbf{A} = (t_{ij})_{S\times S}$ a matrix of zeroes and ones. We identify S with $\mathbf{N}$ and induce an order on S. Let $X = \{x \in S^{\mathbf{N}\cup\{0\}}\colon \forall i\ t_{x_i x_{i+1}} = 1\}$ and $T\colon X \to X$ be the left shift $(Tx)_i = x_{i+1}$. Fix $r \in (0,1)$ and set $t(x,y) = \inf\{i\colon x_i \neq y_i\}$. We endow X with the topology induced by the metric $d_r(x,y) = r^{t(x,y)}$. The **cylinder sets**

$$[\underline{a}] = [a_0, \ldots, a_{n-1}] = \{x \in X\colon \forall i\ x_i = a_i\}$$

form a base for this topology and generate the corresponding Borel σ-algebra $\mathcal{B}$. Let α be the partition $\{[a]\colon a \in S\}$. The elements of α are called **partition sets**,

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and the members of α_0^{n-1} are called cylinders of **length** n. We denote the length of a cylinder $[\underline{a}]$ by $|\underline{a}|$.

X is called **topologically mixing** if (X,T) is topologically mixing. This means that $\forall a,b \in S\ \exists N_{ab}\ \forall n > N_{ab}\ [a] \cap T^{-n}[b] \neq \emptyset$. Throughout this paper, a function ϕ: $X \to \mathbf{R}$ is called **locally Hölder continuous** (with parameter r), if it is uniformly Lipschitz continuous with respect to d_r on all cylinders of length 2. This is equivalent to the requirement that $\exists A > 0, r \in (0,1)$ such that $\forall n \geq 2$ $V_n[\phi] < Ar^n$ where $V_n[\phi] = \sup\{|\phi(x) - \phi(y)|: x_0 = y_0, \ldots, x_{n-1} = y_{n-1}\}$. This notion of Hölder continuity extends the one considered in [18], where $V_n[\phi] < Ar^n$ was also assumed for $n = 1$. Indeed, every function of the form $\phi = \phi(x_0, x_1)$ is locally Hölder continuous, even when $V_1(\phi) = \infty$ (in which case it does not satisfy the condition used in [18]). A close reading of [18] shows that the seemingly greater generality does not affect the arguments in sections 1–4 there.

The **Ruelle Operator** [15] is given by $(L_\phi f)(x) = \sum_{Ty=x} e^{\phi(y)} f(y)$. If $\|L_\phi 1\|_\infty < \infty$ this is a bounded linear operator on the Banach space of bounded continuous functions on X. Note that for a countable Markov shift the sum which defines L_ϕ may be infinite, in which case ϕ must be unbounded in order for it to converge. This is not a problem since local Hölder continuity on a non-compact space does not imply boundness.

In this paper the term 'measure' refers to any σ-finite Borel measure μ which is not trivial in the sense that there is some $A \in \mathcal{B}$ for which $\mu(A) > 0$. We use the notation $\mu(f)$ for the integral of the function f with respect to μ, when it exists. The measure $\mu \circ T$ is the measure given on cylinders by

$$(\mu \circ T)(A) = \sum_{a \in S} \mu(T(A \cap [a])). \tag{1}$$

Integrals with respect to $\mu \circ T$ are given by

$$\int f \, d\mu \circ T = \sum_{a \in S} \int_{T[a]} f(ax) \, d\mu(x).$$

If μ is non-singular (i.e. $\mu \sim \mu \circ T^{-1}$) then $\mu \ll \mu \circ T$ and the function $g_\mu = d\mu / d\mu \circ T$ is well defined $\mu \circ T$ almost everywhere. It is characterized $mod\, \mu \circ T$ by the property that $L_{\log g_\mu}$ acts as the transfer operator of μ, i.e. $\mu(\varphi_1 L_{\log g} \varphi_2) = \mu(\varphi_1 \circ T \cdot \varphi_2)$ for every $\varphi_1 \in L^\infty(\mu)$, $\varphi_2 \in L^1(\mu)$. We will also make use of the measures $\mu \circ T^n$ defined by induction by $\mu \circ T^n = (\mu \circ T^{n-1}) \circ T$.

For every $a \in S$, $n \in \mathbf{N}$ set $Z_n(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x)$ where $\phi_n = \sum_{k=0}^{n-1} \phi \circ T^k$. It was shown in [18] that if X is topologically mixing and ϕ is locally Hölder continuous then the limit

$$P_G(\phi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\phi, a)$$

exists, is independent of a and belongs to $(-\infty, \infty]$. If $\|L_\phi 1\|_\infty < \infty$, this limit is finite and satisfies

$$P_G(\phi) = \sup\left\{h_\mu(T) + \int \phi d\mu \colon \mu \in \mathcal{P}_T(X), \mu(-\phi) < \infty\right\} \tag{2}$$

where $\mathcal{P}_T(X)$ denotes the set of all invariant Borel probability measures. $P_G(\phi)$ is called the **Gurevic Pressure** of ϕ, and is a generalization of the Gurevic topological entropy (Gurevic [7]). (The above results were stated in [18] only for locally Hölder continuous functions for which $V_1(\phi) < \infty$ but the proofs only require that $\sum_{n\geq 2} V_n(\phi)$ be finite.)

In [18] a necessary and sufficient condition was given for Ruelle's Perron–Frobenius theorem to hold: there exist a positive number λ, a positive continuous function h and a σ-finite Borel measure ν such that $L_\phi h = \lambda h$, $L_\phi^* \nu = \lambda \nu$, $\int h d\nu = 1$ and such that for every cylinder $[\underline{a}]$, $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} h\nu[\underline{a}]$ uniformly on compacts. If this is the case, $P_G(\phi) = \log \lambda$ and $dm = h\,d\nu$ is an invariant probability measure which can be interpreted as the 'equilibrium' measure of ϕ in a certain sense (see [18] for details).

In this paper we study the case when Ruelle's Perron–Frobenius theorem fails. The main theme of this work is that the phenomenology of this situation is analogous to that one encounters in the case of a null recurrent or a transient probabilistic Markov chain (see [6], [10], [20]). In this situation $\lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} 0$, but there may exist constants $a_n \nearrow \infty$ for which for every cylinder $a_n^{-1} \sum_{k=1}^n \lambda^{-n} L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} h\nu[\underline{a}]$ pointwise where $L_\phi h = \lambda h, L_\phi^* \nu = \lambda \nu$, $\int h\,d\nu = \infty$. In this case, the measure $dm = hd\nu$ is an infinite invariant measure which can be described as the appropriate 'equilibrium measure' of ϕ. Given ν, the construction of h is done using the techniques of [3] (see also [2], [12], [21], [22], [28], [29], [30], [31]). The main point of this paper is the construction of a conformal measure ν with respect to which these methods can be applied.

We proceed to make our statements more precise. Set

$$Z_n(\phi, a) = \sum_{\substack{T^n x = x \\ x_0 = a}} e^{\phi_n(x)}; \quad Z_n^*(\phi, a) = \sum_{\substack{T^n x = x \\ x_0 = a; x_1, \ldots, x_{n-1} \neq a}} e^{\phi_n(x)}.$$

We introduce the following definition, in analogy with the theory of Markov chains:

Definition 1: Let X be topologically mixing and ϕ be locally Hölder continuous with finite Gurevic pressure $\log\lambda$. ϕ is called:

1. **recurrent** if for some (hence all) $a \in S$, $\sum_{n<\infty} \lambda^{-n} Z_n(\phi, a) = \infty$; and **transient** otherwise;
2. **positive recurrent** if it is recurrent and for some (hence all) $a \in S$, $\sum_{n<\infty} n\lambda^{-n} Z_n^*(\phi, a) < \infty$;
3. **null recurrent** if it is recurrent and for some (hence all) $a \in S$, $\sum_{n<\infty} n\lambda^{-n} Z_n^*(\phi, a) = \infty$.

The notion of positive recurrence was given a different, though equivalent, definition in [18]. The equivalence follows from Theorem 1 below. It can be easily verified that if $\phi = \phi(x_0, x_1)$ then recurrence, positive recurrence and null recurrence are equivalent to the matrix $(e^{\phi(i,j)})_{S\times S}$ being R-recurrent, R-positive and R-null in the terminology of Vere-Jones [24], [24]. The main results of this paper are contained in the following theorem:

THEOREM 1: *Let X be topologically mixing and ϕ locally Hölder continuous with finite Gurevic pressure. ϕ is recurrent iff there exist $\lambda > 0$, a conservative measure ν, finite and positive on cylinders, and a positive continuous function h such that $L_\phi^*\nu = \lambda\nu$ and $L_\phi h = \lambda h$. In this case $\lambda = \exp P_G(\phi)$ and $\exists a_n \nearrow \infty$ such that for every cylinder $[\underline{a}]$ and $x \in X$*

$$\frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k}(L_\phi^k 1_{[\underline{a}]})(x) \underset{n\to\infty}{\longrightarrow} h(x)\nu[\underline{a}], \tag{3}$$

where $\{a_n\}_n$ satisfies $a_n \sim (\int_{[a]} h\,d\nu)^{-1}\sum_{k=1}^{n}\lambda^{-k}Z_k(\phi,a)$ for every $a \in S$. Furthermore,

1. *if ϕ is positive recurrent then $\nu(h) < \infty$, $a_n \sim n \cdot const$, and for every $[\underline{a}]$, $\lambda^{-n}L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} h\nu[\underline{a}]/\nu(h)$ uniformly on compacts;*
2. *if ϕ is null recurrent then $\nu(h) = \infty$, $a_n = o(n)$, and for every $[\underline{a}]$, $\lambda^{-n}L_\phi^n 1_{[\underline{a}]} \underset{n\to\infty}{\longrightarrow} 0$ uniformly on cylinders.*

Remark 1: In the case when ϕ depends on a finite number of coordinates, this theorem can be derived from the work of Vere-Jones on countable matrices ([24], [25]). The case when ϕ depends on an infinite number of coordinates, however, requires techniques which are essentially different. The main new ingredient in the proof is a tightness argument (see Proposition 2).

Remark 2: It follows from the proof that $\log h$ and $\log h \circ T$ are both locally Hölder continuous (in particular h is uniformly bounded away from zero and infinity on partition sets). It follows from (3) that ν and h are uniquely determined up to a multiplicative factor.

Remark 3: The measure $dm = h\,d\nu$ is invariant and conservative, and its transfer operator is given by $\hat{T}f = \lambda^{-1}h^{-1}L_\phi(hf)$. It follows from local Hölder continuity and results in [3] that dm is exact, pointwise dual ergodic and that for dm, every cylinder $[\underline{a}]$ is a Darling–Kac set with an exponential continued fraction mixing return time process. See [2], [3] for definitions and a survey of limit theorems for such measures m.

We now show how to formulate the results of Theorem 1 in terms of suitable **dynamical zeta functions**.

Assume that X is topologically mixing and that ϕ is locally Hölder continuous such that $\|L_\phi 1\|_\infty < \infty$. In this case, by the results of [18], $P_G(\phi)$ is finite and (2) holds. Recall that Ruelle's dynamical zeta function [15] is given by

$$\zeta(t) = \exp\Big(\sum_{n=1}^{\infty} \frac{t^n}{n} Z_n(\phi)\Big)$$

where $Z_n(\phi) = \sum_{a\in S} Z_n(\phi, a) = \sum_{T^n x = x} e^{\phi_n(x)}$. The radius of convergence of ζ is equal to $e^{-P(\phi)}$ where $P(\phi) = \overline{\lim}_{n\to\infty}(1/n)\log Z_n(\phi)$.

If S is finite, $P(\phi) = P_G(\phi)$ whence ζ is holomorphic in $[|z| < e^{-P}]$, where $P = \sup\{h_\mu + \mu(\phi)\}$ (in this case X is compact, so ϕ is bounded and the condition $\mu(-\phi) < \infty$ in (2) is empty). It is also known that in this case ζ has a simple pole in e^{-P} [15].

If S is infinite $P(\phi)$ *may* be strictly larger than P (for examples in the case $\phi = 0$ see [7] and [16]). Therefore, the disc of convergence of ζ may be strictly smaller than $\{z\colon |z| < e^{-P}\}$. We are naturally led to the consideration of the following **local dynamical zeta functions** defined for each $a \in S$,

$$\zeta_a(t) = \exp\Big(\sum_{n=1}^{\infty} \frac{t^n}{n} Z_n(\phi, a)\Big).$$

Note that at least formally, $\zeta = \prod_{a\in S}\zeta_a$. The radius of convergence of ζ_a is independent of a, and is equal to $e^{-P_G(\phi)}$ where $P_G(\phi)$ satisfies (2). Obviously, ζ_a has a singularity in $e^{-P_G(\phi)}$.

As the following corollary shows, the behavior of ζ_a near this singularity determines the recurrence properties of ϕ (this is similar to the role of generating functions in renewal theory [6]). The following corollary is obtained from Theorem 1.

COROLLARY 1: *Let X be topologically mixing and ϕ locally Hölder continuous such that $\|L_\phi 1\|_\infty < \infty$. Fix $a \in S$ and let $R = e^{-P_G(\phi)}$ be the radius of convergence of ζ_a.*

1. *ϕ is recurrent iff $(\log \zeta_a)'(R) = \infty$. In this case, if $dm = hd\nu$ is the corresponding invariant measure and $\{a_n\}_n$ is a return sequence of m, then*

$$(\log \zeta_a)'(t) \sim \frac{m[a]}{R}\left(1 - \frac{t}{R}\right)\sum_{n=1}^{\infty} a_n R^{-n} t^n \quad \text{as } t \nearrow R.$$

2. *ϕ is positive recurrent iff there exists $C_a > 0$ such that $(\log \zeta_a)' \sim C_a(1 - t/R)^{-1}$ as $t \nearrow R$. In this case $C_a = e^{P_G(\phi)} m[a]$ where m is the equilibrium probability measure of ϕ.*
3. *ϕ is null recurrent iff $(\log \zeta_a)' = o(1/(1 - t/R))$ as $t \nearrow R$ and ϕ is recurrent.*

It follows from the corollary that in the positive recurrent case

$$\zeta_a(t) = \left(\frac{1}{1 - e^{P_G(\phi)} t}\right)^{m[a](1+o(1))} \quad \text{as } t \nearrow e^{-P_G(\phi)}$$

where m is the equilibrium *probability* measure of ϕ. If S is finite, we retrieve the well known property of $\zeta = \prod_{a \in S} \zeta_a$ that

$$\zeta_a(t) = \left(1 - e^{P_G(\phi)} t\right)^{-(1+o(1))} \quad \text{as } t \nearrow e^{-P_G(\phi)}$$

(in fact $e^{-P_G(\phi)}$ is a simple pole [15]). In broad terms, the degree of singularity for the full zeta function is distributed among the various local zeta functions according to the equilibrium measure.

In section 2 we apply Theorem 1 to the theory of equilibrium states by describing the measure $dm = h\,d\nu$ as an equilibrium measure in a certain weak sense, when it is infinite. Section 3 contains a proof of Theorem 1.

Notational Convention: We use the following short-hand notation for double inequalities: $\forall a, b > 0$, $B > 1$, $a = B^{\pm 1} b \Leftrightarrow B^{-1} b \le a \le Bb$. We write $a = A^{\pm 1} B^{\pm 1} b$ for $a = (AB)^{\pm 1} b$, and $a = A^{\pm k} b$ for $a = (A^k)^{\pm 1} b$.

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2. Application to the theory of equilibrium states

Let X be topologically mixing and ϕ be a locally Hölder continuous function with finite Gurevic pressure. Assume ϕ is recurrent. Let λ, ν and h denote the eigenvalue, eigenmeasure and eigenfunction given by Theorem 1. It is easy to verify that the measure $dm = h\,d\nu$ is an invariant conservative measure. This is a *Gibbs measure* for ϕ in the following sense: $\forall a, b \in S\ \exists M_{ab} > 1$ such that for m-almost all $x \in X$

$$(4)\qquad m(x_0, \ldots, x_{n-1} | x_n, x_{n+1}, \ldots) = \frac{h(x)e^{\phi_n(x)}}{\lambda^n h(T^n x)} = M_{x_0, x_n}^{\pm 1} e^{\phi_n(x) - nP_G(\phi)}.$$

This is weaker than the Gibbs property used by Bowen in [5], because the bound M_{x_0,x_n} may depend on x. To prove (4), check that the transfer operator of m is given by $\hat{T}f = \lambda^{-1}h^{-1}L_\phi(hf)$ and that $\mathbf{E}_m(f|T^{-n}\mathcal{B}) = (\hat{T}^n f) \circ T^n$. The rest follows by direct computation from the fact that h is bounded away from zero and infinity on partition sets. Note that if ϕ is null recurrent, m is infinite.

We want to describe the measure m as a solution of a suitable variational problem. This was done for the positive recurrent case in [18] so we focus on null recurrent potentials. For such potentials m is infinite and the notion of entropy requires explanation.

We recall the definition given in [11], following the approach of [1]. Let $(X, \mathcal{B}, \mu, T)$ be an ergodic probability preserving transformation. For every measurable set with positive measure A one can define the **induced transformation** T_A: $A \to A$ by $T_A x = T^{\varphi_A(x)} x$ where $\varphi_A(x) = \inf\{n \geq 1: T^n x \in A\}$ (the Poincaré Recurrence theorem guarantees that $\varphi_A < \infty$ almost everywhere on A). It is known that the probability measure $\mu_A(E) = \mu(E \cap A)/\mu(A)$ is T_A-invariant and ergodic, and that its entropy is given by the **Abramov Formula** [4]:

$$h_\mu(T) = \mu(A) h_{\mu_A}(T_A).$$

If μ is infinite, ergodic and conservative, the same method of inducing applies (in this case Poincaré's theorem is replaced by the conservativity assumption). Applying the Abramov formula to T_A, T_B as induced versions of $T_{A\cup B}$ one sees that

$$0 < \mu(A), \mu(B) < \infty \Rightarrow \mu(A)h_{\mu_A}(T_A) = \mu(B)h_{\mu_B}(T_B).$$

Thus, the number $\mu(A)h_{\mu_A}(T_A)$ is independent of the choice of $A \in \mathcal{B}$ (as long as $0 < \mu(A) < \infty$) and may therefore be used as the *definition* of the entropy of the infinite conservative ergodic measure μ.

Example 1 (Krengel [11]): Let (p_{ij}) be a null recurrent irreducible stochastic matrix and (p_i) its stationary vector. Let μ be the corresponding invariant infinite Markovian measure. Then $h_\mu = -\sum_{s,t} p_s p_{st} \log p_{st}$.

For examples arising from interval maps, see [21].

THEOREM 2: *Let X be topologically mixing and ϕ a recurrent locally Hölder continuous function with finite Gurevic pressure.*

1. *For every conservative ergodic invariant measure μ which is finite on partition sets, if $\mu(P_G(\phi) - \phi) < \infty$ then $h_\mu(T) \le \mu(P_G(\phi) - \phi)$.*
2. *Let h and ν be as in Theorem 1 and set $dm = h\,d\nu$. If $m(P_G(\phi) - \phi) < \infty$ then $h_m(T) = m(P_G(\phi) - \phi)$.*

Proof: Without loss of generality assume that $P_G(\phi) = 0$ (we can always pass to the potential $\phi - P_G(\phi)$). Fix some invariant measure μ which satisfies the assumptions of the theorem and choose some partition set A of (finite) positive measure.

Let μ_A be the probability measure $\mu_A(E) = \mu(A \cap E)/\mu(A)$. Let T_A: $A \to A$ be the induced map $T_A x = T^{\varphi_A(x)}x$ where $\varphi_A(x) = 1_A \inf\{n > 0\colon T^n x \in A\}$. Then μ_A is T_A invariant. Let

$$\overline{S}\colon = \{[\underline{a}] \subseteq A\colon\ A \text{ appears only once in } \underline{a} \text{ and } [\underline{a}, A] \neq \emptyset\}.$$

This is a generating Markov partition for T_A ($\mu_A(\cup\overline{S}) = 1$ by conservativity). Set $\overline{X} = (\overline{S})^{\mathbf{N}\cup\{0\}}$ and let π: $\overline{X} \to A \subseteq X$ be the natural injection $\pi([\underline{a}]_1[\underline{a}]_2\ldots) = (\underline{a}_1;\underline{a}_2;\ldots)$. For every μ as in the above set $\overline{\mu} = \mu_A \circ \pi$. It is easy to check that the map π: $\overline{X} \to X$ is a measure theoretic isomorphism between the systems $(A, \mathcal{B}\cap A, \mu_A, T_A)$ and $(\overline{X}, \mathcal{B}(\overline{X}), \overline{\mu}, \overline{T})$ where $\overline{T}$: $\overline{X} \to \overline{X}$ is the left shift. Let $\overline{\phi}$: $\overline{X} \to \mathbf{R}$ be the induced version of the potential ϕ given by

$$\overline{\phi} = \left(\sum_{i=0}^{\varphi_A - 1} \phi \circ T^i\right) \circ \pi.$$

This is a locally Hölder continuous function (in fact, it even satisfies $V_1(\overline{\phi}) < \infty$, since if $x_0 = [\underline{a}] \in \overline{S}$ then $\pi(x) \in [\underline{a}, A]$). The proof of local Hölder continuity is standard, and is therefore omitted.

Let $L_{\overline{\phi}}$ denote the Ruelle operator of $\overline{\phi}$, $L_{\overline{\phi}} f = \sum_{\overline{T}y=x} e^{\overline{\phi}(y)} f(y)$. Set $\overline{\nu} = \nu \circ \pi$ and $\overline{h} = h \circ \pi$. We claim that $L_{\overline{\phi}}^* \overline{\nu} = \overline{\nu}$, $L_{\overline{\phi}} \overline{h} = \overline{h}$. To see this note that

$$\log \frac{dm}{dm \circ T} = \phi + \log h - \log h \circ T$$

(because $f \mapsto h^{-1}L_\phi(hf)$ acts as the transfer operator of m). Let m_A denote the normalized restriction of m to A and $\overline{m} = m_A \circ \pi$. Then since $T_A = T^{\varphi_A}$,

$$\log \frac{dm_A}{dm_A \circ T_A} = \sum_{i=0}^{\varphi_A - 1} \phi \circ T^i + \log h - \log h \circ T_A$$

whence

$$\log \frac{d\overline{m}}{d\overline{m} \circ \overline{T}} = \overline{\phi} + \log \overline{h} - \log \overline{h} \circ \overline{T}. \tag{5}$$

Since m is T invariant, m_A is T_A invariant. It follows that $\overline{m}$ is $\overline{T}$ invariant, whence $L_{\log \overline{g}} 1 = 1$ where $\overline{g} = \log d\overline{m}/d\overline{m} \circ \overline{T}$. It follows from (5) that

$$\sum_{\overline{T}y = x} e^{(\overline{\phi} + \log \overline{h} - \log \overline{h} \circ \overline{T})(y)} = 1$$

whence $L_{\overline{\phi}} \overline{h} = \overline{h}$. We show that $L_{\overline{\phi}}^* \overline{\nu} = \overline{\nu}$. Without loss of generality, $d\overline{\nu} = \overline{h}^{-1} d\overline{m}$ (the only difference is a normalizing constant). Using (5) and the fact that $L_{\log \overline{g}}$ acts as the transfer operator of $\overline{m}$, we have that for every $f \in L^1(\overline{\nu})$,

$$\int L_{\overline{\phi}} f \, d\overline{\nu} = \int \overline{h}^{-1} L_{\overline{\phi}} f \, d\overline{m} = \int L_{\log \overline{g}}(\overline{h}^{-1} f) \, d\overline{m} = \int f \, d\overline{\nu}$$

as required.

It follows from Theorem 1 and the relations $L_{\overline{\phi}} \overline{h} = \overline{h}$, $L_{\overline{\phi}}^* \overline{\nu} = \overline{\nu}$ and $\overline{\nu}(\overline{h}) = \nu(1_A h) < \infty$ that $\overline{\phi}$ is positive recurrent and that $P_G(\overline{\phi}) = 0$. Since $\overline{h} = h \circ \pi$ and $\pi(X) \subseteq A$, $\overline{h}$ is uniformly bounded away from zero and infinity. It follows that $\|L_{\overline{\phi}} 1\|_\infty < \infty$. By (2),

$$\sup \left\{ h_\mu(\overline{T}) + \int \overline{\phi} \, d\mu \colon \mu \text{ is } \overline{T} \text{ invariant}, \ \mu(\overline{X}) = 1, \ \mu(-\overline{\phi}) < \infty \right\} = P_G(\overline{\phi}) = 0.$$

Since for every conservative invariant (possibly infinite) ergodic measure μ such that $\mu(A) < \infty$ and $\mu(-\phi) < \infty$ the measure $\overline{\mu} = \mu_A \circ \pi$ is a $\overline{T}$ invariant ergodic probability measure such that

$$\mu(A)\overline{\mu}(-\overline{\phi}) = -\int_A \sum_{k=0}^{\varphi_A - 1} \phi \circ T^k \, d\mu = \mu(-\phi) < \infty,$$

we have that $h_\mu(T) + \mu(\phi) = \mu(A)[h_{\overline{\mu}}(\overline{T}) + \overline{\mu}(\overline{\phi})] \leq 0$.

We now assume that $\mu = m$ and that $m(-\phi) < \infty$, and show that $h_m(T) + m(\phi) = 0$. $\overline{X}$ clearly satisfies the big images property: $\exists b_1, \ldots, b_N \in \overline{S}$ such that for every $a \in \overline{S}$ there is some b_i such that $[a, b_i]$ is not empty (in fact for every $a, b \in \overline{S}$ $[a, b]$ is non-empty). Since $\overline{h}$ is uniformly bounded away from zero and infinity, $\overline{m}$ is a Gibbs measure for $\overline{\phi}$ in the sense of Bowen [5]: there is some *global* constant $M > 1$ such that for every $\underline{a}_0, \ldots, \underline{a}_{n-1} \in \overline{S}$ and $x \in [\underline{a}_0, \ldots, \underline{a}_{n-1}] \subseteq \overline{X}$,

$$\overline{m}[\underline{a}_0, \ldots, \underline{a}_{n-1}] = M^{\pm 1} \exp \sum_{k=0}^{n-1} \overline{\phi}(\overline{T}^k x) \tag{6}$$

(see [18], Theorem 8). Let $\overline{\alpha} = \{[\underline{a}] \colon \underline{a} \in \overline{S}\}$ denote the natural partition of $\overline{X}$. By the continuity properties of $\overline{\phi}$ and by (6)

$$\begin{aligned} H_{\overline{m}}(\overline{\alpha}) &= - \sum_{[\underline{a}] \in \overline{\alpha}} \overline{m}[\underline{a}] \log \overline{m}[\underline{a}] \\ &\le - \sum_{[\underline{a}] \in \overline{\alpha}} \overline{m}[\underline{a}] \frac{1}{\overline{m}[\underline{a}]} \int_{[\underline{a}]} \overline{\phi} \, d\overline{m} + \log M \\ &= - \int_{\overline{X}} \overline{\phi} \, d\overline{m} + \log M \\ &= - \frac{1}{m(A)} \int_A \sum_{k=0}^{\varphi_A - 1} \phi \circ T^k \, dm + \log M \\ &= - \frac{1}{m(A)} \int \phi \, dm + \log M, \end{aligned}$$

whence $H_{\overline{m}}(\overline{\alpha}) < \infty$. Since $\overline{\alpha}$ is a generator with finite entropy, we have by the Rohlin formula [14] that

$$h_{\overline{m}}(\overline{T}) = - \int \log \frac{d\overline{m}}{d\overline{m} \circ \overline{T}} \, d\overline{m} = - \int \overline{\phi} \, d\overline{m} = - \frac{1}{m(A)} \int \phi \, dm.$$

Multiplying both sides by $m(A)$ we have that $h_m(T) = -m(\phi)$ as required. ∎

Remark 4: It follows from the proof that m is the unique up to a constant conservative ergodic invariant measure such that $H_{\overline{m}}(\overline{\alpha}) < \infty$ and $h_m(T) = m(P_G(\phi) - \phi)$, since by a trivial generalization of an argument of Bowen, if there exists a probability measure which is Gibbs in the sense of Bowen, with a generator which has finite entropy, then this measure is the unique solution of the variational problem (see [5]).

The problem with the last theorem is that frequently both $h_m(T)$ and $m(P_G(\phi) - \phi)$ are infinite. In this situation, the sum $h_m(T) + m(\phi - P_G(\phi))$ is meaningless. The following theorem completes our discussion by treating this case as well.

Set

$$I_\mu = -\sum_{a\in S} 1_{[a]} \log \mu([a]|T^{-1}\mathcal{B}).$$

This is well defined for every μ which is finite on partition sets. The following theorem generalizes theorem 7 in [18] (see also [13], [27], [28]).

THEOREM 3: *Let X be topologically mixing and ϕ locally Hölder continuous with finite Gurevic pressure. Assume that ϕ is recurrent, let h and ν be as in Theorem 1 and set*

$$\phi' = \phi + \log h - \log h \circ T.$$

Then for every conservative invariant measure μ which is finite on partition sets, $I_\mu + \phi' - P_G(\phi')$ is one-sided integrable with respect to μ and

$$-\infty \le \int (I_\mu + \phi' - P_G(\phi'))\, d\mu \le 0; \tag{7}$$

if $\mu \sim \mu \circ T$, the integral in (7) is equal to zero iff μ is proportional to $h\, d\nu$.

Proof: Fix a conservative invariant measure μ finite on partition sets and set

$$g_\mu = d\mu/d\mu \circ T,$$

where $\mu \circ T$ is given by (1). Recall that the transfer operator of μ is given by $L_{\log g_\mu}$ and that

$$\mathbf{E}_\mu(f|T^{-1}\mathcal{B}) = (L_{\log g_\mu} f) \circ T.$$

It follows that

$$I_\mu = -\log g_\mu.$$

Set $g = \lambda^{-1}e^{\phi}h/h \circ T$ where $\lambda = \exp P_G(\phi)$. One checks that $\sum_{Ty=x} g(y) = 1$ and that $\sum_{Ty=x} g_\mu(y) = 1$ for μ almost all $x \in X$ (the first equality follows from the equation $L_\phi h = \lambda h$; the second follows from the identity

$$\mu(f \sum_{Ty=x} g_\mu(y)) = \mu(L_{\log g_\mu}(f \circ T)) = \mu(f),$$

which is satisfied for every $f \in L^1(\mu)$).

We show that $I_\mu + \phi' - P_G(\phi')$ is one-sided integrable. We use the notation $\psi^+ = \psi 1_{[\psi>0]}$ and show that $(I_\mu + \phi' - P_G(\phi'))^+$ is integrable. Fix a sequence of measurable sets $A_n \nearrow X$ such that $0 < \mu(A_n) < \infty$. Fix an arbitrary integrable function $f \ge 0$. Set

$$A_{s,t,n} = A_n \cap [s < g/g_\mu < t].$$

Using the inequality $\log x \le x-1$ we see that for every s,t,n,

$$\begin{aligned}\int_{A_{s,t,n}}(I_\mu+\phi'-P_G(\phi'))^+ f\circ T d\mu &= \int(-\log g_\mu+\log g)^+ 1_{A_{s,t,n}} f\circ T\,d\mu\\ &= \int[\log(g/g_\mu)]^+ 1_{A_{s,t,n}} f\circ T\,d\mu\\ &\le \int\Big(\frac{g}{g_\mu}-1\Big)^+ 1_{A_{s,t,n}} f\circ T d\mu\\ &= \int f\circ T\cdot \mathbf{E}_\mu\Big(\Big(\frac{g}{g_\mu}-1\Big)^+ 1_{A_{s,t,n}}\Big| T^{-1}\mathcal{B}\Big)\,d\mu\\ &= \int f\circ T\sum_{Ty=Tx} g_\mu(y)1_{A_{s,t,n}}(y)\Big(\frac{g(y)}{g_\mu(y)}-1\Big)^+\,d\mu\\ &= \int f\circ T\sum_{Ty=Tx} 1_{A_{s,t,n}}(y)[g(y)-g_\mu(y)]^+\,d\mu.\end{aligned}$$

The last integrand is bounded by $f\circ T$. Since this is true for all s,t,n the integral $\mu[(I_\mu+\phi'-P_G(\phi'))^+]$ is finite. This implies that $I_\mu+\phi'-P_G(\phi')$ is one-sided integrable. Applying the same calculation to $I_\mu+\phi'-P_G(\phi')$ rather than $(I_\mu+\phi'-P_G(\phi'))^+$ yields the inequality

$$\int_{A_{s,t,n}} f\circ T(I_\mu+\phi'-P_G(\phi'))\,d\mu \le \int f\circ T\sum_{Ty=Tx} 1_{A_{s,t,n}}(y)[g(y)-g_\mu(y)]\,d\mu.$$

The integrand on the left is bounded in absolute value by the integrable function $f\circ T$. Its pointwise limit when $s\to 0$, $t,n\to\infty$ is zero, because $\sum_{Ty=Tx}[g(y)-g_\mu(y)]=0$. We may therefore apply the dominated convergence theorem and deduce

$$\int f\circ T[I_\mu+\phi'-P_G(\phi')]\,d\mu\le 0.$$

Since f was arbitrary, (7) follows.

Assume that $\mu\sim\mu\circ T$. We show that the integral in (7) is equal to zero if and only if $d\mu$ is proportional to $h\,d\nu$. If $d\mu$ is proportional to $h\,d\nu$ the integrand in (7) is identically zero because then $I_\mu=-\log g$, where $g=\lambda^{-1}e^\phi h/h\circ T$ (this follows from the fact that the transfer operator of any measure proportional to $h\,d\nu$ is given by $f\mapsto \lambda^{-1}h^{-1}L_\phi(hf)$). We show the reverse implication. Assume that μ is such that $\mu\sim\mu\circ T$ and that there is an equality in (7). A close inspection of the proof shows that this is possible only if $\log(g/g_\mu)=(g/g_\mu)-1$ μ almost everywhere. This is possible only if $g_\mu=g$ *mod* μ. Since $\mu\sim\mu\circ T$, this implies that $g_\mu=g$ *mod* $\mu\circ T$. It follows that $L_{\log g}$ is the transfer operator of μ. Consider the function $\psi = \log g = \phi + \log h - \log h \circ T - \log \lambda$. This is a locally Hölder continuous function (because by Remark 2 after Theorem 1, $\log h$ and $\log h \circ T$ are both locally Hölder continuous). It is also clear that $L_\psi 1 = 1, L_\psi^* \mu = \mu$ whence ψ is recurrent. Since it is also true that $L_\psi^*(hd\nu) = L_{\log g}^*(hd\nu) = hd\nu$ we have by the convergence part of Theorem 1 that μ and $hd\nu$ are proportional. ∎

3. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. Throughout the proof we assume that X is a topologically mixing countable Markov shift and that $\phi: X \to \mathbf{R}$ is locally Hölder continuous with finite Gurevic pressure. Set

$$B_k = \exp \sum_{n=k+1}^{\infty} V_n(\phi) \quad (k = 1, 2, \ldots).$$

Local Hölder continuity implies that $\forall n \geq 1$, $B_n < \infty$ and $B_n \searrow 1$. The following inequality will be used constantly:

$$(8) \qquad x_0 = y_0, \ldots, x_{n-1} = y_{n-1} \Rightarrow \forall m \leq n-1 \quad (e^{\phi_m(x)} = B_{n-m}^{\pm 1} e^{\phi_m(y)}).$$

A frequently used corollary is that $\forall x_a \in [a]$,

$$Z_n(\phi, a) = B_1^{\pm 1}(L_\phi^n 1_{[a]})(x_a).$$

The reader should note that the assumption that the Gurevic pressure is finite implies that *all* of the $Z_n(\phi, a)$ are finite (because by local Hölder continuity $\exists C > 1$ such that $\forall m, n$, $C^{-m} Z_n(\phi, a)^m < Z_{mn}(\phi, a)$). This assumption also implies that the L_ϕ^n are all defined on bounded functions supported inside a finite union of partition sets.

3.1 Existence of ν.

PROPOSITION 1: *If there exists $\lambda > 0$ and a conservative σ-finite measure ν which is finite on some cylinder such that $L_\phi^* \nu = \lambda \nu$ then ϕ is recurrent and $\lambda = e^{P_G(\phi)}$.*

Proof: Choose a cylinder $[\underline{b}]$ with finite positive measure. It is easy to verify that $\lambda^{-1} L_\phi$ acts as the transfer operator of ν, whence by conservativity $\sum_{n \geq 1} \lambda^{-n} L_\phi^n 1_{[\underline{b}]} = \infty$ ν-a.e. on $[\underline{b}]$ (see [2]). Thus, for ν-almost all $x \in [\underline{b}]$

$$\sum_{n=1}^{\infty} \lambda^{-n} Z_n(\phi, b_0) \geq B_1^{-1} \sum_{n=1}^{\infty} \lambda^{-n} (L_\phi^n 1_{[\underline{b}]})(x) = \infty.$$

We show that $\lambda = e^{P_G(\phi)}$. It follows from what we have just proved that $\lambda \le e^{P_G(\phi)}$ because the radius of convergence of the series $\sum_{k\ge1} Z_k(\phi, b_0)x^k$ is $e^{-P_G(\phi)}$. Consider $Z_n(\phi,\underline{b}) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\underline{b}]}(x)$. By local Hölder continuity,

$$\lambda^{-n} Z_n(\phi,\underline{b}) \le B_1 \left[\frac{1}{\nu[\underline{b}]}\int_{[\underline{b}]} (\lambda^{-n} L_\phi^n 1_{[\underline{b}]})\, d\nu\right] \le B_1.$$

By topological mixing and local Hölder continuity $n^{-1}\log Z_n(\phi,\underline{b}) \to P_G(\phi)$, whence $\lambda \ge e^{P_G(\phi)}$. ∎

PROPOSITION 2: *If ϕ is recurrent there exist $\lambda > 0$ and a conservative measure ν, finite and positive on cylinders, such that $L_\phi^*\nu = \lambda\nu$.*

Proof: Fix $a \in S$, set $\lambda = e^{P_G(\phi)}$ and let $a_n = \sum_{k=1}^n \lambda^{-k} Z_k(\phi, a)$. For every $y \in X$ let δ_y denote the probability measure concentrated on $\{y\}$. Fix a periodic point $x_a \in [a]$ and set for every $b \in S$

$$\nu_n^b = \frac{1}{a_n}\sum_{k=1}^n \lambda^{-k} \sum_{T^k y = x_a} e^{\phi_k(y)} 1_{[b]}(y)\delta_y.$$

Clearly $\nu_n^b(X) = \nu_n^b([b]) = \frac{1}{a_n}\sum_{k=1}^n \lambda^{-k}(L_\phi^k 1_{[b]})(x_a)$. It follows from local Hölder continuity, topological mixing and the definition of the Gurevic pressure that for every $b \in S$

$$0 < \varliminf_{n\to\infty} \nu_n^b(X) \le \varlimsup_{n\to\infty} \nu_n^b(X) < \infty.$$

(It is enough to show that $a_n^{-1}\sum_{k=1}^n \lambda^{-k} Z_n(\phi, b)$ is bounded away from zero and infinity for every b. To see this note that $\exists C, c$ such that $Z_n(\phi,b) < CZ_{n+c}(\phi,a)$ and that $\forall k$ $\lambda^{-k}Z_k(\phi,a) < 2B_1$. The last inequality follows from $\lambda^{-km}Z_{km}(\phi,a) > B_1^{-m}(\lambda^{-k}Z_k(\phi,a))^m$.)

We show how to choose a subsequence $\{m_k\}_{k\ge1}$ such that for every $b \in S$, $\{\nu_{m_k}^b\}$ is w^* convergent, and show that the non-trivial measure ν given by $\nu_{m_k}^b \xrightarrow{w^*} \nu|_{[b]}$ satisfies $L_\phi^*\nu = \lambda\nu$. Since X is not compact, to do this we have to prove that $\{\nu_{m_k}^b\}_{k\ge1}$ are all **tight**, i.e.,

$$\forall b\, \forall\varepsilon > 0\, \exists F = F_{b,\varepsilon} \text{ compact such that } \forall n\ \nu_n^b(F^c) < \varepsilon.$$

It follows from the topological mixing of X that if $\{\nu_n^b\}_{n\ge1}$ is tight for some b, then it is tight for *every* b. Therefore, we may restrict ourselves to the case $b = a$ and set $\nu_n^a = \nu_n$.

STEP 1: We show that $\sum_{k\geq 1}\lambda^{-k}Z_k^*(\phi,a)<\infty$. To see this, set $T(x)=1+\sum_{k\geq 1}Z_k(\phi,a)x^k$ and $R(x)=\sum_{k\geq 1}Z_k^*(\phi,a)x^k$. It is not difficult to verify that $\forall x\in(0,\lambda^{-1})$, $T(x)-1=B_1^{\pm 2}R(x)T(x)$. Therefore $\forall x\in(0,\lambda^{-1})$, $R(x)\leq B_1^2$ whence $R(\lambda^{-1})<\infty$.

STEP 2: Set

$$\tau_1(x)=\begin{cases}\inf\{n\geq 1\colon T^n x\in[a]\}, & x\in[a]\\ 0, & x\notin[a]\end{cases} \tag{9}$$

where $\inf\emptyset=\infty$. Define by induction $\tau_n(x)=\tau_1(T^{\tau_1(x)+\cdots+\tau_{n-1}(x)}x)$ if $\tau_{n-1}(x)<\infty$ and $\tau_n(x)=\infty$ else. Note that $\tau_n>0$ only if $x_0=a$. For every sequence of natural numbers $\{n_i\}_{i\geq 1}$ set

$$R(\{n_i\})=\{x\in[a]\colon\forall i\ \tau_i(x)\leq n_i\}.$$

We show that $\forall\varepsilon>0\ \exists\{n_i\}$ such that $\forall n\ \nu_n(R\{n_i\}^c)<\varepsilon$. To see this set

$$Z^*_{k_1,\ldots,k_m}=\sum\{e^{\phi_{k_1+\cdots+k_m}(x)}\colon x_0=a;T^{k_1+\cdots+k_m}x=x;\forall j\leq m\ \tau_j(x)=k_j\}.$$

For $\{n_i\}_{i\geq 1}$ s.t. n_i is larger than the period of x_a,

$$\begin{aligned}\nu_n(R\{n_i\}^c)&\leq\sum_{i=1}^{\infty}\nu_n[\tau_i>n_i]\\
&=\sum_{i=1}^{\infty}\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}\sum_{\substack{T^ky=x_a\\ y_0=a}}e^{\phi_k(y)}1_{[\tau_i>n_i]}(y)\\
&\leq\sum_{i=1}^{\infty}\frac{1}{a_n}\sum_{k=n_i+1}^{n}\lambda^{-k}\sum_{\substack{T^ky=x_a\\ y_0=a}}\sum_{\substack{k_1+\cdots+k_N=k\\ k_i>n_i,N\leq k}}e^{\phi_k(y)}1_{[\forall j\leq N\ \tau_j(y)=k_j]}(y)\\
&\leq B_1^3\sum_{i=1}^{\infty}\frac{1}{a_n}\sum_{k=n_i+1}^{n}\lambda^{-k}\sum_{\substack{k_1+\cdots+k_N=k\\ k_i>n_i,N\leq k}}Z^*_{k_1,\ldots,k_{i-1}}Z^*_{k_i}Z^*_{k_{i+1},\ldots,k_N}\\
&\leq\frac{B_1^3}{a_n}\sum_{i=1}^{\infty}\sum_{k_i=n_i+1}^{\infty}\lambda^{-k_i}Z^*_{k_i}\sum_{k=k_i}^{n}\lambda^{-(k-k_i)}\sum_{\substack{k_1+\ldots+k_N=k\\ N\leq k}}Z^*_{k_1,\ldots,k_{i-1}}Z^*_{k_{i+1},\ldots,k_N}\\
&\leq B_1^5\sum_{i=1}^{\infty}\sum_{k_i=n_i+1}^{\infty}\lambda^{-k_i}Z^*_{k_i}\left(\frac{1}{a_n}\sum_{k=k_i}^{n}\lambda^{-(k-k_i)}Z_{k-k_i}(\phi,a)\right)\\
&\leq B_1^5\sum_{i=1}^{\infty}\sum_{k_i=n_i+1}^{\infty}\lambda^{-k_i}Z^*_{k_i}.\end{aligned}$$

It remains to apply the previous step and choose n_i such that

$$\sum_{k_i=n_i+1}^{\infty} \lambda^{-k_i} Z^*_{k_i} < \frac{\varepsilon}{2^i B_1^5}.$$

STEP 3: Fix $\{n_i\}_i$ such that $\forall n$

$$\nu_n(R\{n_i\}^c) < \varepsilon.$$

For every sequence of natural numbers $\{k_i\}$ set

$$S(\{k_i\}) = \{x \in [a] \colon \forall i\ \tau_i(x) = k_i\}.$$

We show that for every $\varepsilon > 0$ there exists a compact set $F \subseteq [a]$ such that

$$\forall i\ k_i \le n_i \Rightarrow \forall n\ \nu_n(F^c \cap S\{k_i\}) \le \varepsilon \nu_n(S\{k_i\}). \tag{10}$$

This is enough to prove tightness, because (10) implies that for every n,

$$\nu_n(F^c) \le \varepsilon(1 + \nu_n(R))$$

and we already know that the total mass of ν_n is uniformly bounded from above. The F we will construct will be of the form

$$F = \{x \in [a] \colon \forall i\ x_i \le N_i\}$$

where $N_i \in S$ (we are using an order on S induced by the identification $S \approx \mathbf{N}$). Clearly, this is a compact set. We show how to choose $\{N_i\}$. Set

$$Z^*_k(N) = \sum \{e^{\phi_k(x)} \colon x \in [a]; T^k x = x\ ;\ \tau_1(x) = k; \exists i\ x_i > N\}.$$

Obviously, $Z^*_k(N) \searrow 0$ as $n \to \infty$. For every i, we choose N_i in a way such that for every $k \le n_i$

$$Z^*_k(N_i) \le \frac{\varepsilon}{2^i B_1^7} Z^*_k.$$

We make sure that $\{N_i\}$ are chosen in an increasing way and that

$$N_1 > \sup_{i \ge 0}\{x_a(i)\}$$

(recall that x_a was chosen to be periodic, so its coordinates are bounded).

Fix $\{k_i\} \leq \{n_i\}$ such that $\nu_n(S\{k_i\}) > 0$. Fix $N = N(n, \{k_i\})$ such that $k_1 + \cdots + k_N \geq n$. Since $N_i > \sup\{x_a(i)\} \geq a$,

$$\begin{aligned}
\nu_n(F^c \cap S\{k_i\}) &\leq \sum_{i=1}^{N} \nu_n\left\{x \in S\{k_i\}: \exists j \in \left(\sum_{m=1}^{i-1} k_m, \sum_{m=1}^{i} k_m\right) x_j > N_j\right\} \\
&\leq \sum_{i=1}^{N} \nu_n\left\{x \in S\{k_i\}: \exists j \in \left(\sum_{m=1}^{i-1} k_m, \sum_{m=1}^{i} k_m\right) x_j > N_i\right\} \\
&\leq B_1^3 \sum_{i=1}^{N} \frac{1}{a_n} \sum_{l=i}^{N} \lambda^{-(k_1+\cdots+k_l)} Z^*_{k_1,\ldots,k_{i-1}} Z^*_{k_i}(N_i) \\
&\qquad\qquad Z^*_{k_{i+1},\ldots,k_l} 1_{s(\{k_j\}_{j>l})}(x_a) \\
&\leq B_1^6 \sum_{i=1}^{N} \frac{\varepsilon}{2^i B_1^7} \left(\frac{1}{a_n} \sum_{l=i}^{N} \lambda^{-(k_1+\cdots+k_l)} Z^*_{k_1,\ldots,k_l} 1_{s(\{k_j\}_{j=l+1}^{\infty})}(x_a)\right) \\
&\leq \varepsilon\, \nu_n(S\{k_i\}).
\end{aligned}$$

Tightness is proved.

By tightness, there exists a subsequence m_k such that $\forall b \in S$, $\{\nu^b_{m_k}\}_{k\geq 1}$ is w^*-convergent. We denote its limit by ν^b and set $\nu = \sum_{b\in S} \nu^b$. It is not difficult to check that

$$\forall [\underline{b}]\ 0 < \nu[\underline{b}] < \infty. \tag{11}$$

We show that $L^*_\phi \nu = \lambda \nu$. By recurrence, $a_n \nearrow \infty$. A standard calculation shows that for every $[\underline{b}]$ and N, $\nu(1_{[x_0<N]} L_\phi 1_{[\underline{b}]}) = \lambda \nu(1_{[x_1<N]} 1_{[\underline{b}]})$. It follows from the Lebesgue monotone convergence theorem that $\nu(L_\phi 1_{[\underline{b}]}) = \lambda \nu[\underline{b}]$. Since $[\underline{b}]$ was arbitrary, we have that $L^*_\phi \nu = \lambda \nu$.

We show that ν is conservative. One checks that the transfer operator of ν is $\hat{T} = \lambda^{-1} L_\phi$. To prove conservativity it is enough to show that for some positive integrable function f, $\sum_{k\geq 1} \hat{T}^k f = \infty$ almost everywhere. Set $f = \sum_{a\in S} f_a 1_{[a]}$ where $f_a > 0$ are chosen so that $\nu(f) < \infty$. For every $a \in S$ and $x \in [a]$

$$\sum_{k=1}^{\infty} \lambda^{-k} (L^k_\phi f)(x) \geq B_1^{-1} f_a \sum_{k=1}^{\infty} \lambda^{-k} Z_k(\phi, a) = \infty.$$

Conservativity follows. ∎

3.2 THE SCHWEIGER PROPERTY. Let X be a topological Markov shift and μ be a measure supported on X such that $\mu \sim \mu \circ T^{-1}$ and $\mu \sim \mu \circ T$. μ is said to have the **Schweiger property** (see [3]) if there exists a collection of cylinders $\mathcal{R}$ such that:

1. the members of $\mathcal{R}$ have finite positive measures and $\cup\mathcal{R} = X \bmod \nu$;
2. for every $[\underline{b}] \in \mathcal{R}$ and arbitrary cylinder $[\underline{a}]$, if $[\underline{a}, \underline{b}] \neq \emptyset$ then $[\underline{a}, \underline{b}] \in \mathcal{R}$;
3. there exists a constant $C > 1$ such that for every $[\underline{b}] \in \mathcal{R}$ of length n and $\mu \times \mu$ almost all $x, y \in [\underline{b}] \times [\underline{b}]$

$$\frac{d\mu}{d\mu \circ T^n}\Big|_{[\underline{b}]}(x) = C^{\pm 1} \frac{d\mu}{d\mu \circ T^n}\Big|_{[\underline{b}]}(y). \tag{12}$$

Aaronson, Denker and Urbanski proved in [3] that if μ has the Schweiger property, is supported on a topologically mixing topological Markov shift, and is conservative, then:

1. μ is exact (hence ergodic);
2. there exists a σ-finite invariat measure $m \sim \mu$ such that $\log(\frac{dm}{d\nu})$ is bounded on every $B \in \mathcal{R}$;
3. every $[\underline{b}] \in \mathcal{R}$ is a Darling–Kac set for m with a continued fraction mixing return time process (see [3] for definitions and implications);
4. m is **pointwise dual ergodic**: there exist $a_n > 0$ such that for every $f \in L^1(m)$
$$\frac{1}{a_n} \sum_{k=1}^{n} \hat{T}^k f \underset{n\to\infty}{\longrightarrow} m(f) \text{ a.e.,}$$
where $\hat{T}$ is the transfer operator of m.

Rényi's property states that (12) holds for *all* cylinders (see [2]). It follows from local Hölder continuity that ν satisfies Rényi's property with respect to the partition generated by cylinders of length two. It is not true in general, however, that ν satisfies this property with respect to all cylinders, including those of length one (see Example 2 below). In order to obtain information on cylinders of length one as well, we need the following lemma, which was inspired by [3]. For every $c \in S$ set $\mathcal{R}_c = \{[b_0, \ldots, b_{n-1}] : n \in \mathbf{N}, b_{n-1} = c\}$. Note that $[c] \in \mathcal{R}_c$.

LEMMA 1: *Let X be topologically mixing and ϕ locally Hölder continuous. Suppose that ν is a conservative measure, finite and positive on cylinders such that $L_\phi^* \nu = \lambda \nu$. Then $\forall c \in S$ there exists a density function $q = q^{(c)}\colon X \to (0, \infty)$ such that $d\nu_c = q^{(c)}\, d\nu$ has the Schweiger property with respect to $\mathcal{R}_c$. q can be chosen to be constant on partition sets.*

Proof: For every $1 \leq m \leq n-1$ and $[\underline{b}]$ of length n set

$$\phi_m(\underline{b}) = \inf\{\phi_m(x)\colon x \in [\underline{b}]\}.$$

By (8), $\forall x \in [\underline{b}]$ $\phi_m(x) = \phi_m(x_0, \ldots, x_{n-1}) \pm \log B_{n-m}$. Set $q(x) = q^{(c)}(x) = q_{x_0}$ where

$$q_b = \begin{cases} e^{\phi(c,b)}, & [b] \subseteq T[c] \\ 1, & \text{else} \end{cases}$$

and set $d\nu_c = q d\nu$. A calculation shows that $d\nu_c \circ T^n = q_c \circ T^n d\nu \circ T^n$ whence

$$\frac{d\nu_c}{d\nu_c \circ T^n} = \frac{q_c}{q_c \circ T^n} \lambda^{-n} e^{\phi_n}.$$

It follows that for every $x \in [b_0, \ldots, b_{n-1}]$ such that $b_{n-1} = c$

$$\frac{d\nu_c}{d\nu_c \circ T^n}(x) = \frac{q_{b_0}}{q_{x_n}} e^{\phi_n(x)} = B_1^{\pm 1} \lambda^{-n} q_{b_0} e^{\phi_{n-1}(x)}.$$

Thus (12) is proved.

Obviously for every $[\underline{b}]$ in $\mathcal{R}_c$ and for every $[\underline{a}]$, $[\underline{a}, \underline{b}]$ is either empty or in $\mathcal{R}_c$. We show that $X = \bigcup \mathcal{R}_c (\text{mod}\, \nu_c)$. Assume this were not the case. Then $\exists a \in S$ $\exists A \subseteq [a]$ measurable of positive measure such that $\nu_c(A \cap \bigcup \mathcal{R}_c) = 0$. By topological mixing there exists a $[\underline{c}] \subseteq [c]$ such that $[\underline{c}, a] \neq \emptyset$. Choose such a $\underline{c}$ of minimal length. Set $[\underline{c}, A] = [\underline{c}] \cap T^{-|\underline{c}|} A$ where $|\underline{c}|$ denotes the length of $[\underline{c}]$. Then $[\underline{c}, A] \neq \emptyset$ and

$$\int_{[\underline{c}, A]} \frac{d\nu_c \circ T^{|\underline{c}|}}{d\nu_c} d\nu_c = \nu_c(A) > 0$$

whence $\nu_c[\underline{c}, A] > 0$. Since $|\underline{c}|$ is minimal,

$$[\underline{c}, A] \subseteq [c] \backslash T^{-1}(\cup \mathcal{R}_c) = [c] \backslash \bigcup_{n \geq 1} T^{-n}[c],$$

so by conservativity $\nu_c[\underline{c}, A] = 0$. ∎

Example 2: Set $S = \{a, b, 1, 2, 3, \ldots\}$ and $\mathbf{A} = (t_{ij})_{S \times S}$, where $t_{ij} = 1$ if and only if $i \in \{a, b\}$, $j \in \mathbf{N}$ or $i \in \{a, b\}$, $j = i$ or $i = 1$, $j \in \{a, b\}$ or $i \neq a, b, 1$ and $j = i - 1$. Set $\phi(x) = \log p_{x_0, x_1}$ where $p_{aa} = p_{bb} = f_0$ and for all $i \in \mathbf{N}$ and $j \in S$, $p_{ai} = f_i$, $p_{bi} = f'_i$, $p_{ij} = 1$ where f_i, and f'_i will be determined later. Then $Z^*_{n+1}(\phi, 1) = \sum_{k=0}^{n-1} (f_{n-k} + f'_{n-k}) f_0^k$ and

$$Z_n(\phi, 1) = Z^*_n(\phi, 1) + \sum_{k=1}^{n-1} Z^*_{n-k}(\phi, 1) Z_k(\phi, k).$$

Now choose $f_0 = 1/4$, $f_i = C/2^i$ and $f'_i = C/4^i$, where $C > 0$ is a constant such that $\sum_{n \geq 1} Z^*_k(\phi, 1) = 1$. It follows from the renewal theorem that $Z_n(\phi, 1)$ tends to $1/\sum_{n \geq 1} n Z^*_n(\phi, 1) > 0$ as n tends to infinity. Thus $P_G(\phi) = 0$ and ϕ

is positive recurrent. Let ν be the corresponding eigenmeasure (the existence of which is guaranteed by Proposition 2). Then there is no density vector $\{p_k\}$ such that the resulting measure satisfies Rényi's condition because such a vector must satisfy $p_k \asymp p_{ak}, p_{bk}$ whereas $p_{ak} \not\asymp p_{bk}$.

3.3 Existence of h and $\{a_n\}_n$.

Proposition 3: *If ϕ is recurrent then $\exists h > 0$ and $\exists \{a_n\}_{n=1}^{\infty}$ such that $L_\phi h = \lambda h$ and such that for every cylinder $[\underline{b}]$ and $x \in X$*

$$\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}(L_\phi^k 1_{[\underline{b}]})(x) \underset{n\to\infty}{\longrightarrow} h(x)\nu[\underline{b}].$$

Furthermore, h is bounded away from zero and infinity on partition sets, $\log h$ and $\log h \circ T$ are locally Hölder continuous, and every cylinder is a Darling–Kac set for $dm = h\,d\nu$ with a continued fraction mixing return time process.

Proof: Since ϕ is recurrent, there exists a conservative measure ν, finite and positive on cylinders, such that $L_\phi^* \nu = \lambda\nu$. Fix an arbitrary $c \in S$ and set $\mathcal{R}_c = \{[b_0, \ldots, b_{n-1}] \colon n \in \mathbf{N}, b_{n-1} = c\}$. By Lemma 1, $\exists \nu_c \sim \nu$ with the Schweiger property with respect to $\mathcal{R}_c$ such that $d\nu_c/d\nu$ is constant on partition sets. By the results cited in the last section, there exists an exact invariant measure m which is equivalent to ν_c, hence also to ν. Its derivative $dm/d\nu$ is bounded away from zero and infinity on members of $\mathcal{R}_c$ (because $d\nu_c/d\nu$ is constant on partition sets). This measure is pointwise dual ergodic: there exist $a_n > 0$ such that for every $f \in L^1(m)$

$$\frac{1}{a_n}\sum_{k=1}^{n}\hat{T}^k f \underset{n\to\infty}{\longrightarrow} \int f\,dm \text{ a.e.} \tag{13}$$

Set $h = dm/d\nu$. Since ν is equivalent to m and m is exact, ν is conservative ergodic and can only have one invariant density (up to a constant). Thus h and m are independent of c. It also follows from (13) that $\{a_n\}$ is independent of c (up to a constant and asymptotic equivalence). The results of the previous section imply that every member of $\mathcal{R}_c$ is a Darling–Kac set for m with a continued fraction mixing return time process. Since m is independent of c and c is arbitrary, this is true for every member of $\bigcup_{c\in S}\mathcal{R}_c$, i.e. for all cylinders. The same reasoning shows that h is bounded away from zero and infinity on every cylinder. Thus, since ν is positive and finite on cylinders, so is m.

We show that h and $\{a_n\}$ are the required eigenfunction and sequence. The transfer operator of dm is given by $\hat{T}f = \lambda^{-1}h^{-1}L_\phi(hf)$ (because $dm = h\,d\nu$

and the transfer operator of ν is given by $\lambda^{-1}L_\phi$). Thus, for every cylinder $[\underline{b}]$

$$\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}L_\phi^k 1_{[\underline{b}]} = \frac{1}{a_n}h\sum_{k=1}^{n}\hat{T}^k(h^{-1}1_{[\underline{b}]}). \tag{14}$$

For every cylinder $[\underline{b}]$ the function $h^{-1}1_{[\underline{b}]}$ is m-integrable (because h is bounded away from zero on cylinders). Thus (14) implies that for m-almost every $x \in X$ for every cylinder $[\underline{b}]$

$$\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}(L_\phi^k 1_{[\underline{b}]})(x) \underset{n\to\infty}{\longrightarrow} h(x)\nu[\underline{b}]. \tag{15}$$

Since ν is positive on cylinders, and $m \sim \nu$, there is a dense set of points $x \in X$ for which (15) is valid for every cylinder $[\underline{b}]$. By (8), $\forall m \geq 1\ \forall k\ V_m[\log(L_\phi^k 1_{[\underline{b}]})] < \log B_m$ and we have that the logarithm of each of the summands in the left hand side of (15) is uniformly continuous in x. It follows that h has a version for which (15) holds *everywhere* for every cylinder $[\underline{b}]$. This version must satisfy

$$\forall m \geq 1 \quad V_m[\log h] < \log B_m \tag{16}$$

whence $\log h$ and $\log h \circ T$ are locally Hölder continuous. We see, again, that h is uniformly bounded away from zero and infinity on partition sets, because the last estimation is also valid for the case $m = 1$.

It is now possible to show that h is an eigenfunction. Applying L_ϕ on both sides of (15) (and noting that by conservativity $a_n \to \infty$) it is easy to see that $L_\phi h \leq \lambda h$. Set $f = h - \lambda^{-1}L_\phi h$. This is a non-negative function which satisfies $\sum_{k>0}\lambda^{-k}L_\phi^k f < \infty$. Since ν is ergodic conservative with transfer operator $\lambda^{-1}L_\phi$, this is impossible unless $f = 0$ ν-a.e. Since f is continuous and ν supported everywhere, $f = 0$ whence $L_\phi h = \lambda h$. ∎

3.4 Identification of $\{a_n\}_n$.

PROPOSITION 4: *Let m and $\{a_n\}_n$ be as in Proposition 3. Then for every $a \in S$*

$$a_n \sim \frac{1}{m[a]}\sum_{k=1}^{n}\lambda^{-k}Z_n(\phi, a).$$

Proof: Let $\hat{T}$ denote the transfer operator of m. For every cylinder $[\underline{a}]$ of length N set $Z_n(\phi,\underline{a}) = \sum_{T^n x = x} e^{\phi_n(x)}1_{[\underline{a}]}(x)$ and choose some $x_{\underline{a}} \in [\underline{a}]$. By (16), for every $N \geq 1$ and almost all $x_{\underline{a}} \in [\underline{a}]$

$$\lambda^{-n}Z_n(\phi,\underline{a}) = B_N^{\pm 1}(\lambda^{-n}L_\phi^n 1_{[\underline{a}]})(x_{\underline{a}}) = B_N^{\pm 2}(\hat{T}^n 1_{[\underline{a}]})(x_{\underline{a}}). \tag{17}$$

By (13)

$$\underline{\lim}_{n\to\infty}, \overline{\lim}_{n\to\infty} \left[\frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} Z_k(\phi, \underline{a})\right] = B_N^{\pm 2} m[\underline{a}]. \tag{18}$$

The idea is to sum over $[\underline{a}] \subseteq [a]$ and deduce that

$$\underline{\lim}_{n\to\infty}, \overline{\lim}_{n\to\infty} \left[\frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} Z_k(\phi, a)\right] = B_N^{\pm 2} m[a]$$

which implies, since N is arbitrary, that both limits coincide and are equal to $m[a]$. We need a regularity argument to deal with the possibility that there may be an infinite number of $[\underline{a}] \subseteq [a]$ such that $|\underline{a}| = N$.

Let $\varepsilon > 0$ and $F = F_\varepsilon$ be a compact such that $m([a]\setminus F) < \varepsilon$. We denote by $[a] \cap \alpha_0^{N-1}$ the set of all cylinders of length N that are included in $[a]$. Then,

$$\begin{aligned}
\frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} Z_k(\phi, a) =& \frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} \sum_{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}} Z_k(\phi, \underline{a}) \\
=& \frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} \sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1} \\ [\underline{a}]\cap F\neq\emptyset}} Z_k(\phi, \underline{a}) \\
&+ \frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} \sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1} \\ [\underline{a}]\subseteq[a]\setminus F}} Z_k(\phi, \underline{a}).
\end{aligned}$$

Using (16), (17) and the pointwise dual ergodicity of m, we have that for almost every $z_a \in [a]$

$$\begin{aligned}
\frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} \sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1} \\ [\underline{a}]\subseteq[a]\setminus F}} Z_k(\phi, \underline{a}) \leq& B_N^2 \frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} \sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1} \\ [\underline{a}]\subseteq[a]\setminus F}} [h^{-1} L_\phi^k (h 1_{[\underline{a}]})](x_{\underline{a}}) \\
\leq& B_N^2 B_1 \frac{1}{a_n}\sum_{k=1}^{n} [\lambda^{-k} h^{-1} L_\phi^k (h 1_{[a]\setminus F})](z_a) \\
\leq& B_N^2 B_1 \frac{1}{a_n}\sum_{k=1}^{n} (\widehat{T}^k 1_{[a]\setminus F})(z_a) \\
\underset{n\to\infty}{\longrightarrow}& B_N^2 B_1 m([a]\setminus F).
\end{aligned}$$

Thus,

$$\frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} Z_k(\phi, a) = \sum_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1} \\ [\underline{a}]\cap F\neq\emptyset}} \left[\frac{1}{a_n}\sum_{k=1}^{n} \lambda^{-k} Z_k(\phi, \underline{a})\right] + O(\varepsilon).$$

The sum on the right is finite, because F is compact. It follows from this and (18) that

$$\underline{\lim}_{n\to\infty}, \overline{\lim}_{n\to\infty}\left[\frac{1}{a_n}\sum_{k=1}^{n}\lambda^{-k}Z_k(\phi,a)\right] = B_N^{\pm 2}m\left(\bigcup_{\substack{[\underline{a}]\subseteq[a]\cap\alpha_0^{N-1}\\ [\underline{a}]\cap F_\varepsilon\neq\emptyset}}[\underline{a}]\right)+O(\varepsilon).$$

Letting ε tend to zero and then N tend to infinity, we have that the upper and lower limits coincide and are equal to $m[a]$. ∎

3.5 Positive recurrence and Null recurrence. Throughout this subsection we assume that X is topologically mixing, ϕ is locally Hölder continuous and recurrent and that λ, ν and h are its corresponding eigenvalue, eigenmeasure and eigenfunction, respectively. As usual, $dm = h\,d\nu$ and $\hat{T}f = \lambda^{-1}h^{-1}L_\phi(hf)$ is its transfer operator.

Proposition 5: *Under the above assumptions, $\nu(h) < \infty$ iff ϕ is positive recurrent, and $\nu(h) = \infty$ iff ϕ is null recurrent.*

Proof: Fix $a \in S$ and let $\tau_1(x)$ be given by (9). By conservativity, τ_1 is well defined and finite ν-almost everywhere in $[a]$. Set $\psi_N = 1_{[\tau_1=N]}$. By (16), $\forall N\ \forall k > N$

$$(\hat{T}^k\psi_N)1_{[a]} = B_1^{\pm 2}\lambda^{-N}Z_N^*(\phi,a)(\hat{T}^{k-N}1_{[a]})1_{[a]}.$$

Taking limits in both sides, using pointwise dual ergodicity, we see that

$$\lambda^{-N}Z_N^*(\phi,a) = B_1^{\pm 2}m[\tau_1 = N]/m[a].$$

It follows that

$$\sum_{n=1}^{\infty}n\lambda^{-n}Z_n^*(\phi,a) = B_1^{\pm 2}\frac{1}{m[a]}\int_{[a]}\tau_1\,dm.$$

The result follows from the ergodicity and conservativity of m and the Kac formula $\int_{[a]}\tau_1\,dm = m(X)$. ∎

Proposition 6: *Under the above assumptions, for every cylinder $[\underline{a}]$,*

1. *if ϕ is null recurrent then*

$$\lambda^{-n}L_\phi^n 1_{[\underline{a}]}\underset{n\to\infty}{\longrightarrow}0$$

uniformly on cylinders whence $a_n = o(n)$;

2. *if ϕ is positive recurrent then*

$$\lambda^{-n}(L_\phi^n 1_{[\underline{a}]})(x) \underset{n\to\infty}{\longrightarrow} \frac{h(x)}{\nu(h)}\nu[\underline{a}]$$

uniformly on compacts whence $a_n \sim n \cdot const$.

Proof: Assume that ϕ is null recurrent and fix some $a \in S$. Since L_ϕ is positive and h is uniformly bounded away from zero and infinity on $[a]$, it is enough to show that $\lambda^{-n}h^{-1}L_\phi^n(h1_{[a]}) \underset{n\to\infty}{\longrightarrow} 0$ uniformly on cylinders. Choose unions of partition sets F_n such that $F_n \nearrow X$ and $0 < m(F_n) < \infty$. ϕ is null recurrent so $m(F_N) \nearrow \infty$. Set $f_N = 1_{[a]} - 1_{F_N} \cdot m[a]/m(F_N)$. For every $b \in S$ the usual estimations yield (for $\|\cdot\|_1 = \|\cdot\|_{L^1(m)}$)

$$\begin{aligned}\|1_{[b]}\hat{T}^n 1_{[a]}\|_\infty \leq & B_1^3 \frac{1}{m[b]}\|1_{[b]}\hat{T}^n 1_{[a]}\|_1 \\ \leq & \frac{B_1^3}{m[b]}\left(\|1_{[b]}\hat{T}^n f_N\|_1 + \frac{m[a]}{m(F_N)}\|1_{[b]}\hat{T}^n 1_{F_N}\|_1\right) \\ \leq & \frac{B_1^3}{m[b]}\left(\|\hat{T}^n f_N\|_1 + \frac{m[a]m[b]}{m(F_N)}\right).\end{aligned}$$

Here, $\hat{T}$ is the transfer operator of m. Since $m(f_N) = 0$ and m is exact (it is equivalent to ν, and ν has the Schweiger property), it follows from a theorem of M. Lin (see theorem 1.3.3 in [2]) that $\|\hat{T}^n f_N\|_{L^1(m)} \to 0$. It follows from this and from the fact that $m(F_N) \uparrow \infty$ that $\|1_{[b]}\hat{T}^n 1_{[a]}\|_\infty \underset{n\to\infty}{\longrightarrow} 0$ as required.

Assume now that ϕ is positive recurrent. Without loss of generality, assume that $\nu(h) = 1$. For every cylinder $[\underline{a}]$ the family $\{\lambda^{-n}L_\phi^n 1_{[\underline{a}]}\}_n$ is equicontinuous and uniformly bounded on partition sets $[b]$ (by $C\|h1_{[b]}\|_\infty$ where $C = 1/\inf\{h(x)\colon x \in [\underline{a}]\}$). It follows that every subsequence has a subsequence of its own which converges uniformly on compacts. It is enough to show that the only possible limit point is $h\nu[\underline{a}]$, because it will then immediately follow from the equicontinuity of $\{\lambda^{-n}L_\phi^n 1_{[\underline{a}]}\}_n$ that this sequence tends uniformly on compacts to $h\nu[\underline{a}]$.

Assume that $\lambda^{-n_k}L_\phi^{n_k}1_{[\underline{a}]}$ tends to φ pointwise. Since for every k, $\lambda^{-n_k}L_\phi^{n_k}1_{[\underline{a}]} \leq Ch$ and Ch is integrable, we have by the dominated convergence theorem that

$$\begin{aligned}\int |\varphi - h\nu[\underline{a}]|\, d\nu &= \lim_{k\to\infty}\int |\lambda^{-n_k}L_\phi^{n_k}1_{[\underline{a}]} - h\nu[\underline{a}]|\, d\nu \\ &= \lim_{k\to\infty}\int |\hat{T}^{n_k}(h^{-1}1_{[\underline{a}]} - \nu[\underline{a}])|\, dm.\end{aligned}$$

Since m is exact, the last limit is equal to zero and we have that $\varphi = h\nu[\underline{a}]$ almost everywhere. Since φ must be continuous, it must be equal to $h\nu[\underline{a}]$ everywhere. (Note that this argument does not work if ϕ is null recurrent, because in this case $h^{-1}1_{[\underline{a}]} - \nu[\underline{a}]$ is not integrable.) ∎

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